BALANCED CENTRAL NT SCHEMES FOR THE SHALLOW WATER EQUATIONS

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Abstract The numerical method we consider is based on the nonstaggered central scheme proposed by Jiang, Levy, Lin, Osher, and Tadmor (SIAM J. Numer. Anal. 35, 2147(1998)) that was obtained by conversion of the standard central NT scheme to the nonstaggered mesh. The generalization we propose is connected with the numerical evaluation of the geometrical source term. The presented scheme is applied to the nonhomogeneous shallow water system. Including an appropriate numerical treatment for the source term evaluation we obtain the scheme that preserves quiescent steady-state for the shallow water equations exactly. We consider two different approaches that depend on the discretization of the riverbed bottom. The obtained schemes are well balanced and present accurate and robust results in both steady and unsteady flow simulations.

Keywords: balance law, central schemes, exact C-property, shallow water equations.

Introduction

In recent years many numerical schemes have been adopted for application to hyperbolic balance laws. Different schemes are obtained according to the discretization of the source term. In presence of the stiff source terms in balance laws, the implicit evaluation of the source term is needed, since the explicit evaluation can produce numerical instabilities. For other type of bal-

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ance laws, which incorporate the geometrical source terms, such as shallow water equations, an essentially different approach must be used. Here the explicit evaluation of the source term, that additionally accounts for the crucial property of balancing between the flux gradient and the source term, leads to very accurate and robust numerical schemes. One of the first numerical schemes based on that approach was developed by Bermudez and Vazquez ([1, 2]). Their numerical scheme is of finite volume type, with the source term evaluation that includes the upwinding in such a way that the obtained scheme is consistent with the quiescent steady state, i.e., it satisfies the C-property. In [9] the surface gradient method used in combination with the MUSCL Hancock scheme leads to a balanced numerical scheme. The central–upwind schemes ([3]) have been also developed for the shallow water equations. Furthermore, in [7] higher order numerical schemes, i.e., the finite difference ENO and WENO schemes, were extended to the balance laws.

In this work we focus on the nonstaggered central NT scheme ([8, 10]). In [10], the central NT schemes were already developed for the balance laws. However, the approach used there is aimed to the balance laws with a stiff source term, while here we consider systems with a geometrical source term and present a completely different numerical treatment.

The paper is organized as follows. After the nonstaggered central NT scheme for the homogeneous case is presented, its extension to the balance law is given. In second section we apply the extended schemes to the shallow water equations. The discretizations of the source term are made according to the required balancing property. Additionally the numerical scheme must be adapted in such a way that the transformations from the nonstaggered to the staggered values and vice a versa preserve the quiescent flow. In that sense, based on the different riverbed discretizations, we introduce two reformulations of the numerical scheme for the shallow water flow case. In this section we also prove that both reformulations satisfy the exact C-property. On numerical tests in the last section we verify the accuracy of the given schemes and present the improvement obtained by using the balanced version of the schemes.

1. Central NT scheme.

In this section we give a short overview of the central schemes. Detail description of this schemes can be found in [8, 10, 5], etc.

Let us consider the one–dimensional homogeneous hyperbolic conservation law system

$$\partial_t u + \partial_x f(u) = 0. \tag{1}$$

Cells of size Δx, $I_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}], i = 0, \ldots, N$, where $x_{i\pm\frac{1}{2}} = x_i \pm \frac{\Delta x}{2}$ and points $x_i = i\Delta x$ as the *i*th cell center are defined. Furthermore, the staggered cells $[x_i, x_{i+1}]$ are denoted with $I_{i+\frac{1}{2}}$. For a solution $u(x,t)$, $u_i^n =$

$u(x_i, t^n)$ denotes a point value of the solution at $t = t^n$. The abbreviations $\overline{u}_i^n$ and $\overline{u}_{i+\frac{1}{2}}^n$ are used for the average values of the solution over the cells I_i and $I_{i+\frac{1}{2}}$ respectively. We start with the integration of (1) over a control volume $I_{i+\frac{1}{2}} \times [t^n, t^{n+1}]$ and obtain the expression

$$\overline{u}_{i+\frac{1}{2}}^{n+1} = \overline{u}_{i+\frac{1}{2}}^n - \frac{1}{\Delta x}\left[\int_{t^n}^{t^{n+1}} f(u(x_{i+1}, t))dt - \int_{t^n}^{t^{n+1}} f(u(x_i, t))dt\right] \quad (2)$$

The second order Nessyahu–Tadmor central scheme (central NT scheme) is based on a piecewise linear representation of the solution on each grid cell,

$$u(x, t^n) = \sum_i \left(\overline{u}_i^n + u_i'(x - x_i)\right) \chi_{I_i}(x). \quad (3)$$

A slope u_i' inside the cell is computed by using some standard slope limiting procedure ([5]). The simplest choice is a minmod limiter $u_i' = \frac{1}{\Delta x} MM(u_{i+1} - u_i, u_i - u_{i-1})$, where $MM(a, b)$ is the minmod function. Now, $\overline{u}_{i+\frac{1}{2}}^n$ is the cell average at time t^n obtained by integrating the piecewise linear function (3) over the cell $I_{i+\frac{1}{2}}$, i.e.,

$$\overline{u}_{i+\frac{1}{2}}^n = \frac{1}{2}(\overline{u}_i^n + \overline{u}_{i+1}^n) + \frac{\Delta x}{8}(u_i' - u_{i+1}'). \quad (4)$$

Thus, with (4) a second order accuracy in space would be obtained. The approximations of the integrals in (2) such that the second order accuracy in time is attained, yields to the central NT scheme that could be written in the predictor–corrector form as

$$u_i^{n+\frac{1}{2}} = u_i^n - \frac{\Delta t}{2\Delta x} f_i', \; u_i^n = \overline{u}_i^n, \quad (5)$$

$$\overline{u}_{i+\frac{1}{2}}^{n+1} = \overline{u}_{i+\frac{1}{2}}^n - \frac{\Delta t}{\Delta x}\left(f(u_{i+1}^{n+\frac{1}{2}}) - f(u_i^{n+\frac{1}{2}})\right). \quad (6)$$

Here f_i' denotes the spatial derivative of the flux. In order to prevent spurious oscillations in the numerical solution, it is necessary to evaluate the quantity f_i' using a suitable slope limiter ([8]). In that sense the slope limiter procedure can be applied directly to the values $f(\overline{u}_i^n)$ or the relation $f_i' = A(\overline{u}_i^n)\, u_i'$ should be used. In this work the second approach in combination with a minmod slope limiter is chosen.

After the staggered values $\overline{u}_{i+\frac{1}{2}}^{n+1}$ in the corrector step of the scheme are computed, the nonstaggered version of the central NT scheme developed in [8], returns back to the nonstaggered mesh. That means, the average nonstaggered values $\overline{u}_i^{n+1}$ must be determined. In order to do that, first the piecewise linear representation of the form

$$\widetilde{u}(x,t^{n+1}) = \sum_i \left(\overline{u}_{i+\frac{1}{2}}^{n+1} + u'_{i+\frac{1}{2}}(x - x_{i+\frac{1}{2}}) \right) \chi_{I_{i+\frac{1}{2}}}(x). \tag{7}$$

is constructed. The staggered cell derivatives are computed by applying a slope limiter procedure to the staggered values $\overline{u}_{i+\frac{1}{2}}^{n+1}$. The values $\overline{u}_i^{n+1}$ are now obtained by averaging this linear interpolant over the cell I_i,

$$\overline{u}_i^{n+1} = \frac{1}{2}(\overline{u}_{i-\frac{1}{2}}^{n+1} + \overline{u}_{i+\frac{1}{2}}^{n+1}) - \frac{\Delta x}{8}(u'_{i+\frac{1}{2}} - u'_{i-\frac{1}{2}}). \tag{8}$$

We consider now a balance law system

$$\partial_t u + \partial_x f(u) = g(u,x). \tag{9}$$

In order to solve it with the central NT scheme an appropriate extension of the presented scheme should be applied. Several possible approaches are given in [10]. We consider here only the geometrical type source terms, therefore an upwinded discretization will be crucial for obtaining a stable numerical scheme. The additional requirements on the source term evaluation that depend on the particular balance law and that are proposed in the next section, ensure the good accuracy of the numerical scheme developed in this work.

Let us proceed as in the homogeneous case. The integration of (9) over a control volume $I_{i+\frac{1}{2}} \times [t^n, t^{n+1}]$ gives

$$\begin{aligned} \overline{u}_{i+\frac{1}{2}}^{n+1} &= \overline{u}_{i+\frac{1}{2}}^{n} - \frac{1}{\Delta x}\left[\int_{t^n}^{t^{n+1}} f(u(x_{i+1},t))dt - \int_{t^n}^{t^{n+1}} f(u(x_i,t))dt \right] \\ &\quad + \frac{1}{\Delta x}\int_{t^n}^{t^{n+1}} \int_{x_i}^{x_{i+1}} g(u(x,t),x)dxdt. \end{aligned} \tag{10}$$

To obtain a second order scheme, all the integrals in the above expression must be evaluated according to this order. The flux integral is approximated as before by using the midpoint rule, i.e.,

$$\int_{t^n}^{t^{n+1}} f(u(x_i,t))dt \approx \Delta x f(u_i^{n+\frac{1}{2}})$$

where the predictor values $u_i^{n+\frac{1}{2}}$ are now evaluated by using the relation

$$u_i^{n+\frac{1}{2}} = u_i^n + \frac{\Delta t}{2\Delta x}(-f'_i + g_i^n \Delta x) \tag{11}$$

obtained from (9). The term g_i^n can be evaluated pointwise or some other approximation could be applied as we will see in the proceeding of this work. Furthermore, the approximation of the source term integral in (9) is defined such that second order accuracy in time is obtained. With this discretization the corrector step of our scheme

$$\overline{u}_{i+\frac{1}{2}}^{n+1} = \overline{u}_{i+\frac{1}{2}}^{n} - \frac{\Delta t}{\Delta x}\left(f(u_{i+1}^{n+\frac{1}{2}}) - f(u_i^{n+\frac{1}{2}})\right) + \Delta t g(u_i^{n+\frac{1}{2}}, u_{i+1}^{n+\frac{1}{2}}) \tag{12}$$

is obtained. The spatial accuracy depends on the definition of the term

$$g(u_i^{n+\frac{1}{2}}, u_{i+1}^{n+\frac{1}{2}}).$$

Transformations from the staggered values to the nonstaggered ones and in the opposite direction are obtained with the relations (4) and (8) as in the homogeneous case.

2. Balanced central NT scheme for the shallow water equations.

In this section we apply the nonstaggered central NT schemes to the shallow water equations. In the shallow water case (9) is defined with

$$u = \begin{pmatrix} h \\ hv \end{pmatrix}, \; f = \begin{pmatrix} hv \\ hv^2 + \frac{1}{2}gh^2 \end{pmatrix}, \; g = \begin{pmatrix} 0 \\ gh(-\frac{dz}{dx} - \frac{M^2 v|v|}{h^{4/3}}) \end{pmatrix}. \tag{13}$$

Here $h = h(x,t)$ is the water depth, $v = v(x,t)$ is the water velocity, g is acceleration due to gravity, $z = z(x)$ is the bed level, and $M = M(x)$ is the Manning's friction factor.

The crucial property we want to be satisfied when the central NT scheme is applied to the shallow water equations is the exact C-property ([1]). The numerical scheme has exact C-property if it preserves a steady state of the quiescent flow $h + z = const,\ v = 0$ exactly. Since in that case the balancing between the flux gradient and the source term must be obtained, we refer to the scheme developed in this paper as to the balanced central NT scheme. In order to define the central NT scheme for the shallow–water system, the source term g_i^n in the predictor step (11) and the term $g(u_i^{n+\frac{1}{2}}, u_{i+1}^{n+\frac{1}{2}})$ that arises in the corrector step (12) should be determined. From this point on, when the derivations of the variables are evaluated, we use just a minmod limiter function. Following the idea of decomposing the source term, we propose to evaluate g_i^n as

$$g_i^n = g_{i,L}^n + g_{i,R}^n, \tag{14}$$

where

$$g_{i,L}^n = s_i^2 \frac{1 - s_i}{2}\begin{pmatrix} 0 \\ -gh_i^n \frac{z_i - z_{i-1}}{\Delta x} \end{pmatrix}, \; g_{i,R}^n = s_i^2 \frac{1 + s_i}{2}\begin{pmatrix} 0 \\ -gh_i^n \frac{z_{i+1} - z_i}{\Delta x} \end{pmatrix}.$$

The parameter s_i in the ith cell is defined by

$$s_i = \left\{ \begin{array}{rl} -1 & \text{, if } h_i' = h_i^n - h_{i-1}^n \\ 1 & \text{, if } h_i' = h_{i+1}^n - h_i^n \\ 0 & \text{, if } h_i' = 0 \end{array} \right. . \tag{15}$$

Depending on the side that is chosen when the variable and the flux derivations are evaluated the defined parameter changes the sign value. Thus, the expression (14) actually includes the source term upwinding. In this way the source term discretization is made according to the flux gradient evaluation. For the term $g(u_i^{n+\frac{1}{2}}, u_{i+1}^{n+\frac{1}{2}})$ we propose to use just the centered approximation

$$g(u_i^{n+\frac{1}{2}}, u_{i+1}^{n+\frac{1}{2}}) = \left(\begin{array}{c} 0 \\ g\frac{h_i^{n+\frac{1}{2}} + h_{i+1}^{n+\frac{1}{2}}}{2}\left(-\frac{z_{i+1}-z_i}{\Delta x}\right) \end{array} \right). \tag{16}$$

The part of the source term concerning friction forces is omitted in (14) and (16). The reason lies in the fact it does not appear when a quiescent flow case is considered and we evaluate it just pointwise.

Since we want that the defined numerical scheme preserves the quiescent flow exactly, we must first check if the balancing between the flux gradient and the source term is obtained. In the quiescent flow case the variable, the flux and the source term vector reduce to

$$u = \left(\begin{array}{c} h \\ 0 \end{array} \right), \quad f = \left(\begin{array}{c} 0 \\ \frac{1}{2}gh^2 \end{array} \right), \quad g = \left(\begin{array}{c} 0 \\ gh(-\frac{dz}{dx}) \end{array} \right). \tag{17}$$

If we use the definition (14) in (11), it is not hard to see that in the quiescent flow case the equality

$$u_i^{n+\frac{1}{2}} = u_i^n \tag{18}$$

holds. In the similar way from (12) by using (16) we get

$$\overline{u}_{i+\frac{1}{2}}^{n+1} = \overline{u}_{i+\frac{1}{2}}^{n}. \tag{19}$$

The obtained equalities are consequence of balancing in both steps of the numerical scheme. From (18) and (19) we can conclude that in the quiescent flow case no time evolution of the variables occurs. Hence, if the initial discretization satisfies the quiescent flow condition, this condition would be preserved if the procedure of passing form the original to the staggered mesh and vice a versa is defined in an appropriate way. For that purpose the modification of the original nonstaggered version of the central NT scheme for applying it to the shallow water equations is needed. We propose here two different reformulations of the algorithm for evaluation of the staggered and the nonstaggered cell averages in the shallow water case. These reformulations are based on discretizations of the riverbed bottom.

2.1 The interface type reformulation

We consider first the case where the bed topography is defined at the cell interfaces and the bed shape is approximated as a linear function inside each cell. That means, the values $z_{i-\frac{1}{2}}$ and $z_{i+\frac{1}{2}}$ are known, while the height of the riverbed bottom inside the cell I_i is expressed as $z(x) = z_i + \frac{1}{\Delta x}(z_{i+\frac{1}{2}} - z_{i-\frac{1}{2}})(x - x_i)$. At the cell center the relation $z_i = \frac{z_{i-\frac{1}{2}}+z_{i+\frac{1}{2}}}{2}$ is valid.

Now we start with our reformulation. The corrections we propose are connected with the way of evaluation of u_i' and $u'_{i+\frac{1}{2}}$ in (3) and (7). The given reformulation is based on the surface gradient method.

Since in the quiescent flow case the second component in the variable vector is equal zero, the modifications will be done just for the first component, i.e., for the variable h. When the central NT scheme is considered, the water depth and the riverbed bottom are supposed to be linear inside each cell. Here, the linearization of the water depth will be made indirectly by prescribing first the linearization of the water level $H(x)$ and then by using the relation $h(x) = H(x) - z(x)$. The linearization $H(x)$ inside a cell I_i is obtained by using a slope limiting procedure on the cell values $H_i = h_i + z_i$. Thus, for $x \in I_i$ we have $H(x) = H_i + H_i'(x - x_i)$. The derivation of the water depth can be obviously calculated as

$$h_i' = H_i' - \frac{1}{\Delta x}(z_{i+\frac{1}{2}} - z_{i-\frac{1}{2}}). \tag{20}$$

When the staggered values are considered the reformulation is again applied just to h. First we define the point values of the water level on the staggered mesh as

$$\widetilde{H}_{i+\frac{1}{2}} = h_{i+\frac{1}{2}} + \widetilde{z}_{i+\frac{1}{2}}. \tag{21}$$

Here the term $\widetilde{z}_{i+\frac{1}{2}} = z_{i+\frac{1}{2}} - \frac{1}{2}\left(z_{i+\frac{1}{2}} - \frac{z_i+z_{i+1}}{2}\right)$ is the corrected riverbed bottom. The reason of this correction lies in the fact that the riverbed is not linear inside the staggered cell $I_{i+\frac{1}{2}}$. Now the discrete derivatives $H'_{i+\frac{1}{2}}$ are derived from the staggered values $\{\widetilde{H}_{i+\frac{1}{2}}\}$ by using a standard slope limiter procedure. Then the relation

$$h'_{i+\frac{1}{2}} = \widetilde{H}'_{i+\frac{1}{2}} - \frac{1}{\Delta x}(z_{i+1} - z_i) \tag{22}$$

is applied.

We claim that for the described treatment of the cell average evaluations the reformulated nonstaggered central scheme is consistent with the quiescent flow case. Let us prove that.

From the relation (19) follows that the staggered values do not change in the time step of the numerical scheme. That means, it is enough to prove that transformations from the staggered values to the nonstaggered ones and then back return us the same values that we start form. We concentrate just on the variable h.

The quiescent flow at the discrete level can be written as

$$H_i = h_i + z_i = const. \tag{23}$$

From relations (4) and (20) we obtain

$$\begin{aligned} \overline{h}^n_{i+\frac{1}{2}} &= \frac{1}{2}(\overline{h}^n_i + \overline{h}^n_{i+1}) \\ &\quad + \frac{\Delta x}{8}\left(H'_i - \frac{z_{i+\frac{1}{2}} - z_{i-\frac{1}{2}}}{\Delta x} - H'_{i+1} + \frac{z_{i+\frac{3}{2}} - z_{i+\frac{1}{2}}}{\Delta x}\right) \\ &= \frac{1}{2}(\overline{h}^n_i + \overline{h}^n_{i+1}) - \frac{1}{2}\left(z_{i+\frac{1}{2}} - \frac{z_i + z_{i+1}}{2}\right). \end{aligned} \tag{24}$$

The last equality is obtained by using the fact that for the quiescent flow case $H'_i = 0$ and by applying the relations $z_{i+\frac{1}{2}} - z_{i-\frac{1}{2}} = 2(z_{i+\frac{1}{2}} - z_i)$ and $z_{i+\frac{3}{2}} - z_{i+\frac{1}{2}} = 2(z_{i+1} - z_{i+\frac{1}{2}})$. By using (24) in (21) simple calculations give us $\widetilde{H}_{i+\frac{1}{2}} = \frac{1}{2}(h_i + h_{i+1} + z_i + z_{i+1})$ and since (23) is valid, $\widetilde{H}_{i+\frac{1}{2}}$ is constant over the whole domain. Finally, the nonstaggered values are evaluated from (8) by using (22) as

$$\overline{h}^{n+1}_i = \frac{1}{2}(\overline{h}^{n+1}_{i-\frac{1}{2}} + \overline{h}^{n+1}_{i+\frac{1}{2}}) \tag{25}$$

$$- \frac{\Delta x}{8}\left(H'_{i+\frac{1}{2}} - \frac{z_{i+1} - z_i}{\Delta x} - H'_{i-\frac{1}{2}} + \frac{z_i - z_{i-1}}{\Delta x}\right). \tag{26}$$

By taking into account expressions (24) for the staggered values of h and the fact $H'_{i+\frac{1}{2}} = 0$, the right side of (26) reduces to $\overline{h}^n_i$. With this the proof of the consistency with the quiescent flow case is ended.

2.2 The cell centered type reformulation

Now we consider the case in which the bottom heights z_i at cell centers are given. The surface gradient method is then applied in the next way. Let us notice that the term $\overline{u}^n_{i+\frac{1}{2}}$ appears only in relation (11), where the approximation of the spatial part is added to this term. Therefore it is not necessary to evaluate the term $\overline{h}^{n+1}_{i+\frac{1}{2}}$ directly. Instead, we compute the staggered values $\overline{H}^n_{i+\frac{1}{2}}$ at the same way at it was described in the previous paragraph, i.e., by using values $H_i = h_i + z_i$ and a slope limiter procedure for evaluating the derivatives. After the evolution step (12) is applied, we obtain instead of $\overline{h}_{i+\frac{1}{2}}^{n+1}$ as a first component of the variable $\overline{u}_{i+\frac{1}{2}}^{n+1}$, the staggered value of the water level $\overline{H}_{i+\frac{1}{2}}^{n+1}$. The next step of the method, in which the nonstaggered values are computed, give us the water level values $\overline{H}_i^{n+1}$. Finally, by applying the simple relation $\overline{h}_i^{n+1} = \overline{H}_i^{n+1} - z_i$ the water depth values at time step $t = t^{n+1}$ are obtained.

We prove now that the scheme obtained with this reformulation preserves also the quiescent steady state exactly.

Again, as in the previous reformulation, due to equalities (18) and (19), we concentrate just on verification if the procedure of passing from staggered to nonstaggered values and back preserve the water depth in the quiescent flow case. Since (23) is valid $H_i' = 0$, so from (4) we get

$$\overline{H}_{i+\frac{1}{2}}^n = \frac{1}{2}(\overline{H}_i^n + \overline{H}_{i+1}^n) = const.$$

As the staggered values do not change in the evolution step the values $\overline{H}_{i+\frac{1}{2}}^{n+1}$ will be constant and the term ${H'}_{i+\frac{1}{2}}$ will be equal zero. By including the established facts in (8) we have

$$\overline{H}_i^{n+1} = \frac{1}{2}(\overline{H}_{i-\frac{1}{2}}^n + \overline{H}_{i+\frac{1}{2}}^n) = \overline{H}_i^n, \tag{27}$$

so the equality $\overline{h}_i^{n+1} = \overline{h}_i^n$ is obviously fulfilled.

3. Numerical results

In this section we present the improvements obtained by using the proposed balanced versions of the nonstaggered central NT scheme on several test problems. In all the test problems the CFL coefficient is set to 0.5.

3.1 A quiescent steady test

In this test section we are interested in the quiescent steady state preserving property of our scheme. We test it on the problem with the riverbed geometry proposed by the Working Group on Dam–Break Modelling, as described in [2]. The water level is initially defined with $H = 15m$ and the water is at rest. The riverbed and the initial water level are presented in Fig. 1. Computations are performed by using the interface type reformulation and $\Delta x = 7.5m$. In Fig. 2 we can see the performance obtained by using the balanced and the pointwise central NT scheme. The numerical errors that appear when just the pointwise source term evaluation is used are very large, therefore unacceptable for practical use.

3.2 Tidal wave propagation in a channel with a discontinuous bottom

We consider here an unsteady problem taken again from [2]. It is used to establish the correctness of the central NT scheme in the case of a gradually varied flow and to show that the proposed source term evaluation is necessary when a discontinuous bottom is present. The riverbed is the same as in previous test problem. The tidal wave incoming from the left boundary is defined with $h(0,t) = 16.0 - 4.0 \sin\left[\pi\left(\frac{4t}{86400} + \frac{1}{2}\right)\right]$. The water $12m$ high is initially at rest. The right boundary condition is $v(1500,t) = 0$. The computations are performed with the space step $\Delta x = 7.5m$. We give numerical results after $t = 10800s$. Results presented in Fig. 3, where the comparison between the balanced and nonbalanced versions of the central NT schemes is made, clearly illustrate the superiority of the balanced schemes. Then in Fig. 4 the numerically obtained velocity profile is compared with the approximate one (see [1]). We

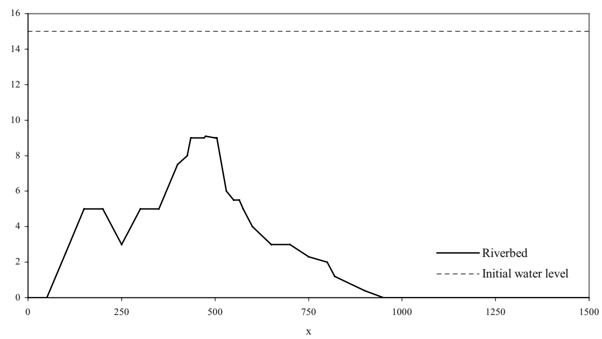

Figure 1. Initial conditions for the test problem 3.1.

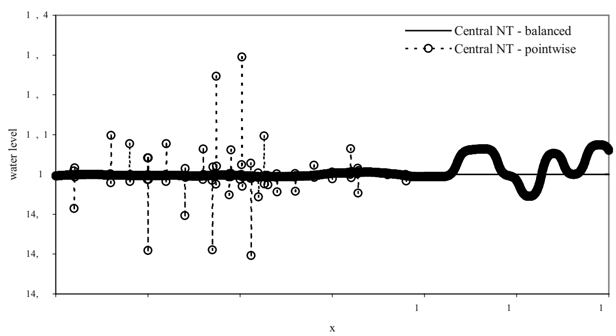

Figure 2. Comparison in water level for the quiescent steady state at $t = 100s$. Test problem 3.1.

can conclude that the agreement is excellent. This suggest that the proposed scheme is accurate for tidal flow over an irregular bed. Such a behaviour could be very encouraging for real water flow simulations over natural watercourses.

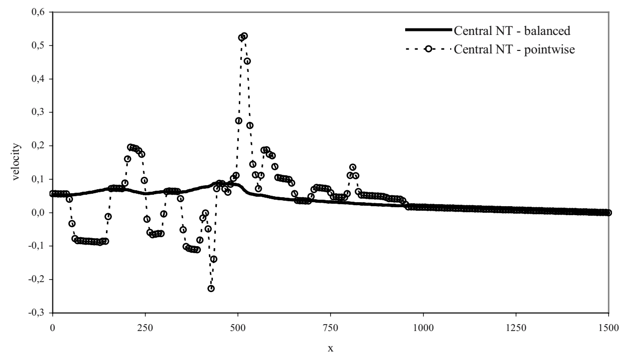

Figure 3. Comparison of velocity at $t = 10800s$ in the test problem 3.2.

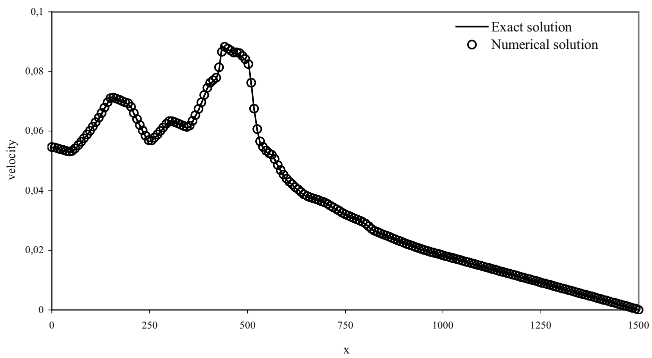

Figure 4. Velocity computed with the balanced central NT scheme vs. asymptotic solution at $t = 10800s$. Test problem 3.2.

3.3 A convergence test over an exponential bump

This is a steady state test problem used for testing the convergence properties of the balanced central NT scheme. We know that a central NT scheme is second order accurate when it is used on homogeneous conservation laws. Now we want to confirm this order of accuracy for the balance laws also. The riverbed bottom is supposed to be given with a smooth function $z(x) = 0.2e^{-\frac{4}{25}(x-10)^2}$. The domain is in the range $[0, 20]$ and the initial condition is steady subcritical flow with a constant discharge equal $4,42m^2/s$. The stationary solution could be evaluated analytically and should be preserved. With the given test problem we examine accuracy and the convergence properties of our scheme. Here we test the interface type reformulation. The convergence test results are presented in Table 1. We can notify that the experimentally established orders coincide very well with the theoretic ones.

Table 1. Accuracy of the central NT scheme. Test problem 3.3.

Errors in water level

N	L_1 error	L_1 order	L_∞ error	L_∞ order
20	3.72×10^{-3}		1.58×10^{-2}	
40	1.18×10^{-3}	1.65	5.34×10^{-3}	1.57
80	2.83×10^{-4}	2.07	1.71×10^{-3}	1.64
160	6.76×10^{-5}	2.07	4.98×10^{-4}	1.78
320	1.66×10^{-5}	2.02	1.29×10^{-4}	1.95

Errors in discharge

N	L_1 error	L_1 order	L_∞ error	L_∞ order
20	6.82×10^{-3}		2.51×10^{-2}	
40	2.06×10^{-3}	1.73	1.10×10^{-2}	1.20
80	5.20×10^{-4}	1.99	3.74×10^{-3}	1.55
160	1.29×10^{-4}	2.01	1.07×10^{-3}	1.81
320	3.21×10^{-5}	2.00	2.76×10^{-4}	1.95

3.4 LeVeque test example over bump

This test problem is suggested by LeVeque ([4]). The bottom topography is defined with

$$z(x) = \begin{cases} 0.25(\cos(10\pi(x-0.5))+1) & \text{, if } |x-0.5| < 0.1 \\ 0 & \text{, otherwise} \end{cases} \tag{28}$$

over the domain $[0,1]$. The initial conditions are

$$v(x,0) = 0 \text{ and } h(x,0) = \begin{cases} 1.01 - z(x) & \text{, if } 0.1 < x < 0.2 \\ 1.0 - z(x) & \text{, otherwise} \end{cases} . \tag{29}$$

As in [4] we take $g = 1$. A small perturbation that is defined with the initial conditions splits into two waves. The left-going wave leaves the domain, while the right-going one moves over the bump. Results are shown at time $t = 0.7s$

after the left-going wave already left the domain while a right-going passes the bump. The computations are performed with a space step $\Delta x = 0.005$ and by using the cell centered type reformulation. The disturbance in the pointwise version caused by the varying riverbed bottom can be clearly seen in Fig. 5. These numerical errors are of the same order as the disturbance that is moving over the domain. That leads to the conclusion that the nonbalanced scheme is especially unfavorable for the cases in which small disturbances appear.

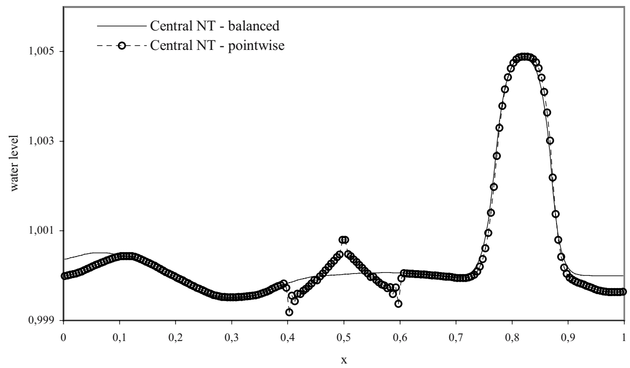

Figure 5. Comparison in water level at $t = 0.7s$. Test problem 3.4.

3.5 Dam–break over a rectangular bump

This is a test problem taken from [7]. The purpose of this test is to check the balanced central NT scheme in the case of rapidly varying flow over a discontinuous bottom. The riverbed is given with

$$z(x) = \begin{cases} 8 & \text{, if } |x - 1500/2| < 1500/8 \\ 0 & \text{, otherwise} \end{cases} , \tag{30}$$

while the initial conditions are

$$H(x,0) = \begin{cases} 20 & \text{, if } x \leq 750 \\ 15 & \text{, otherwise} \end{cases} \quad \text{and} \quad v(x,0) = 0. \tag{31}$$

The Manning friction factor is set to 0.1. The computations are performed with the space step $\Delta x = 2.5m$ and the cell centered type reformulation. In Figs. 6 and 7 we compare the balanced and the nonbalanced central NT scheme at time $t = 15s$. The improvements obtained by using a balanced version are clearly visible.

4. Conclusion remarks

In this paper we present the extension of the nonstaggered central NT schemes to the balanced laws with geometrical source terms. The equilibrium type discretization of the source term, that includes the balancing with the flux gradient is used. The schemes are applied to the shallow water equations. The computations performed on several test problems show very good results in steady and unsteady flow cases.

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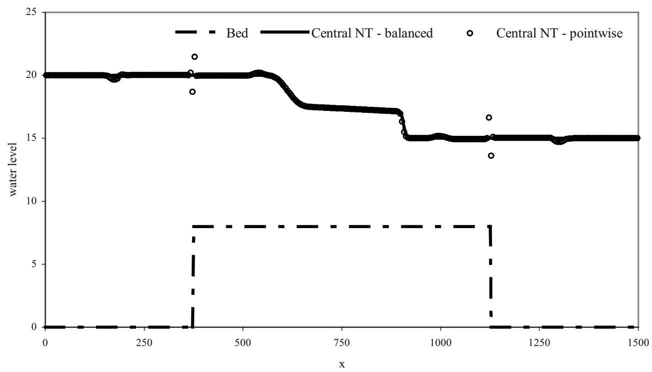

Figure 6. Comparison in water level at $t = 15s$. Test problem 3.5.

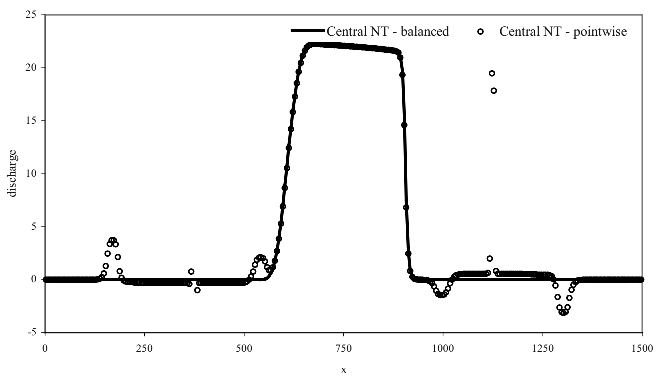

Figure 7. Comparison in discharge at $t = 15s$. Test problem 3.5.

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